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H.J.J. TE RIELE & J.H.C. LISMAN

THE APPORTIONMENT OF REPRESENTATIVES IN THE SECOND CHAMBER OF DUTCH PARLIAMENT

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The apportionment of representatives in the Second Chamber of Dutch Parliament*)

by

H.J.J. te Riele & J.H.C. Lisman**)

ABSTRACT

This report deals with systems to achieve a satisfactory apportionment of representatives in the Second Chamber of Dutch Parliament.

In par. 2 the present system is discussed. It has the well-known fault that the greater parties are favoured as to their representation. Another system without this bias, simple and straightforward, was presented in the Netherlands in 1973 and is described in par. 3.

As a matter of fact the allocation of seats in Parliament according to the vote distribution can be considered a problem of minimizing the inequality between the repartition of votes and that of seats. Then some inequality coefficient should be minimized. In par. 4 we discuss three designs; moreover, a striking difference is shown to exist between the inequality coefficients which are minimized by the systems of par. 2 and 3.

An additional remark is made referring to non-proportional representation.

Finally, in par. 5 we present a system of weighted votes to be used when voting in Parliament. This system corrects perfectly the unfairness in the seat distribution, which is always present to a certain extent.

KEY WORDS & PHRASES: *Apportionment of representatives, minimization*

*) This report has been submitted for publication elsewhere and is not for review.

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PREFACE

This report presents a mathematical study of a problem and possible solutions in a somewhat unexpected domain: politics. It attempts to quantify a number of objections of the authors (and of many others) to the present system of apportionment of representatives in the Second Chamber of Dutch Parliament and it indicates a possibly suitable substitute system. Its presentation is such that interested politicians should be able to trace and understand the main results.

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1. INTRODUCTION

Each four years new representatives in the Second Chamber of Dutch Parliament have to be elected. Political parties then nominate their candidates and the number of votes for these parties has to be translated into the number of representatives, which sum up to 150. This translation is a question of mathematics, under political desires and constraints. If one aims to reach at an apportionment of representatives which fits the vote distribution best, then some optimum procedure may be appropriate. In practice, however, matters are handled differently. Actually, the problem is in the field of Operational Research, but politics and O.R. are no neighbours! As a matter of fact politicians and jurists concerned with public government and law-making are not acquainted with the ideas of O.R. and there is only a modest start in jurimetrics, whereas politicometrics has been hardly born.

In par. 2 we first describe the present system of apportionment of representatives in the Second Chamber of Dutch Parliament, and the main fault of this method. Par. 3 refers to a better system. In par. 4 we turn to other solutions by applying simple optimization techniques. Moreover, a striking difference between certain minimizing properties of the systems of par. 2 and 3 is described. Finally, in par. 5 we present a new refinement in the system of voting in colleges: the weighted vote.

2. THE PRESENT SYSTEM OF APPORTIONMENT

The present system in The Netherlands is called that of proportional representation: political parties are entitled to a number of representatives in the Second Chamber of Parliament, which is proportional to the number of votes they have scored in the country as a whole.

The number of representatives in the Chamber amounts to 150. The total number of valid votes divided by 150 is called the quota (Q). At present there is a so-called voting-threshold, which means that a party should have scored at least this quota in order to receive one or more seats. Now the total number of votes of each party which is admitted is divided by the quota, and the integer parts of the resulting numbers show

how many representatives for each party enter the Chamber. Obviously, a small number of seats is then left to be allocated. This is done as follows. "Successively every time one of the remaining seats is allocated to the party which, after allocation, shows the greatest number of votes per seat" (system of the greatest means, introduced by Hagenbach-Bischoff).

An example may enlighten this procedure. Let there be 5 parties A,B,C,D and E. Suppose there are 20 representatives to be elected. and the total number of votes is 1000, according to Table 1. The quota Q amounts to 1000/20 = 50.

Table 1

Party	Number of votes V	Number of representatives R'	$\frac{V}{R'+1}$	Number of representatives R"	$\frac{V}{R''+1}$	Number of representatives R
A	528	10	48	11	44	11
B	205	4	41	4	41	4
C	180	3	45	3	45	4
D	84	1	42	1	42	1
E	3					
Total	1000	18		19		20

From the total in the third column it appears that two seats are still to be allocated. In the fourth column for each party the number of votes V has been divided by the provisional number of seats (representatives) R' + 1. Party A shows the greatest quotient and the first remaining seat is allocated to this party. Now a provisional distribution R" comes into play (fifth column). The procedure is repeated and it turns out that party C is getting the second remaining seat. Then all 20 seats are allocated (last column).

In the election 1977 the possibility for parties to make combinations was introduced. In this report we leave this complication out of discussion.

The most important and fundamental objection to this system is that it works in favour of the greatest parties. It can be seen from Table 1, but an extreme example in Table 2 makes things still more clear.

Table 2

Party	Number of votes V	Number of representatives R'	$\frac{V}{R'+1}$	Number of representatives R
A	900	18	47.37	19
B	94	1	47.00	1
C	6			
Total	1000	19		20

The proportion of votes for the parties A and B is about 9:1, but the proportion of the representatives for these parties is 19:1! This seems to be unacceptable, whatever political preferences are handled.

We now give a rigorous proof that this system systematically favours the greater parties. Let there be n parties, V_i be the number of votes of the i-th party, $V = \sum_{i=1}^{n} V_i$, and let $V_i \geq V_{i+1}$. Suppose that m ($m \leq n$) parties have scored more than the quota Q ($= V/150$). Initially, R_i' seats are allocated to the i-th party, with $R_i' = \text{entier}(V_i/Q)$ ($i = 1,2,\ldots,m$). The number of remaining votes is denoted by $V_1', V_2', \ldots, V_m'$ ($0 \leq V_i' < Q$). Next, the mean number of votes per seat is computed under the assumption that one more seat would have been allocated than was done initially. We denote these means by g_i ($i = 1,2,\ldots,m$), $g_i = V_i/(R_i'+1)$. The party with the greatest mean receives the first remaining seat. The mean of this last party is now recomputed on the basis of a number of seats again raised by one, and the party which now shows the greatest mean receives the second remaining seat. The procedure is continued until all seats are allocated.

In order to find out how far this system favours the greater parties, we compare g_i with g_j where $i < j$. A simple calculation shows that $g_i > g_j$ if the following inequality is satisfied:

$$V_j' < Q(1 - V_j/V_i) + V_i'(V_j/V_i),$$

in which case party j certainly does not receive the first remaining seat. Whether party i receives this seat or not, depends on the other parties.

As an example we assume that party i has ten times the number of votes of party j, so that $V_j/V_i = 1/10$. Then the above inequality may be put into words as follows: the smaller party j certainly does not receive the first remaining seat if its number of remaining votes V_j' is smaller than 90% of Q, irrespective of the number of remaining votes of the greater party. Now if we assume that the first remaining seat is allocated to party i , then its new mean is $g_i^* = V_i/(R_i' + 2)$. It is easily seen that $g_i^* > g_j$ if

$$V_j' < Q(1 - 2V_j/V_i) + V_i'(V_j/V_i).$$

Hence, the smaller party certainly does not receive the second remaining seat if its number of remaining votes is smaller than 80% of Q, irrespective of the number of remaining votes of the greater party.

So there is a clear bias in favour of the greater parties.

3. THE SYSTEM R.E.

There is another system, called the R.E. system*) which does not show this fault. It has been introduced in the Netherlands in 1973 by LISMAN [3, see also 4]. This system can be described in four points.

i) First of all it is determined which parties are to be admitted in Parliament. Their number of votes is at least equal to the quota Q.

ii) Next, the number of votes of each party is divided by the total number of votes and multiplied by 150. The votes for parties which do not enter Parliament are left out of consideration. The resulting numbers (generally no integers) show a distribution of seats which corresponds

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*) The abbreviation R.E. stands for "Rounded off Exact" (distribution). It should be remarked that WILLCOX [8] and LAGUE [5] have developed another system which, however, leads to the same results as R.E.

exactly to the distribution of the votes. It is called the exact distribution, and the figures sum up to 150

iii) However, there are only integer numbers of representatives, so that we now round off the figures to integers, ≥ 0.5 upwards and < 0.5 downwards. If the rounded numbers sum up to 150, then we have arrived at the desired distribution of seats.

iv) It may happen that the numbers sum up to an integer which is slightly more or less than 150. The difference is rarely more than one or two. In such a case the denominator in step ii) has to be chosen a little bit greater or smaller, such that after rounding off the numbers of representatives sum up to 150. Such a denominator can always be found.

It is an essential feature of the system R.E. that during this procedure the mutual proportions of the votes remain the same. Moreover, the system is free from the problem of remaining seats.

The system R.E. may be demonstrated by the following example (Table 3). We start from the same data as in Table 1. The parties A,B,C and D are admitted. The third column refers to a distribution of seats which corresponds exactly to the numbers of votes recorded to the four parties. The figures are rounded off and they appear to sum up to 21. The division by T in the third column has to be repeated with a somewhat greater denominator (T' = 1010) in order to obtain a sum of 20 seats.

Table 3

Party	Number of votes V	Number of Representatives			
		Exact $\frac{V}{T}$ x20	Rounded off R'	Exact $\frac{V}{T'}$ x20	Rounded off R
A	528	10.59	11	10.46	10
B	205	4.11	4	4.06	4
C	180	3.61	4	3.56	4
D	84	1.69	2	1.66	2
Total	997	20	21		20

It must be emphasized that in this case, with an increase of T in order to obtain a sum of 20, the greater party A -moving downwards- loses one seat. If, however, we would have a decrease of T for this purpose, one of the greater parties -moving upwards- would receive one seat. This demonstrates the neutrality of the system R.E..

As a final remark it may be stated that votes for parties which are not admitted to Parliament are lost. In the present system, however, they flow over to a certain extent to the greater parties. In the R.E.system they have no function at all. In order to create a destination for these lost votes it might be considered appropriate to give the electors the opportunity to present a second vote, referring to a second party if the first is not admitted.

In Table 4 an illustration is given of the elections in 1977. The number of representatives is given according to the present system and to the system R.E. as well. The shift to the greater parties is clear.

Table 4

Party	Number of votes V	Number of Representatives	
		Present system	System R.E.
1	2 183 793	53	52
2	2 655 391	49	49
3	1 492 689	28	27
4	452 423	8	8
5	177 010	3	3
6	143 481	2	3
7	140 910	3	3
8	79 421	1	2
9	77 972	1	1
10	69 914	1	1
11	59 487	1	1
	8 162 491	150	150
others	158 234		
	8 320 725		

We think that politicians of various signature will prefer the distribution according to the system R.E.. For instance: the proportion of the numbers of votes for party 1 and 2 respectively amounts to 1.060. As to the numbers of representatives the proportion amounts to 1.082 for the present system and 1.061 for the system R.E.!

As to the example given in Table 2 the system R.E. leads to 18 representatives for party A and 2 for party B, which is far better than the rather unacceptable result according to the present system.

4. MINIMIZING PROCEDURES

Besides the two systems for allocation of seats in Parliament there are others, but they all are to a certain extent pragmatic.

However, some optimization procedure, designed for this purpose, would be more fundamental. A starting point is that the distribution of seats should fit that of the votes "as close as possible". This asks for minimization of some inequality coefficient, under two constraints: the numbers of representatives are integers, and they sum up to 150.

We shall discuss here three systems which minimize some inequality coefficient. The exact distribution of seats R_i' shall always be given by

$$R_i' = \frac{V_i}{\sum_{i=1}^{n} V_i} \times 150 \quad (i = 1,2,\ldots,n),$$

so that $\sum_{i=1}^{n} R_i' = 150^{*)}$. Generally, of course, the R_i' are no integers. For simplicity, we shall not take into account the voting-threshold (in that case we generally have $\sum R_i' < 150$). It suffices to say here that the voting-threshold does not cause any essential complications with respect to the systems to be discussed here.

a. The first inequality coefficient is the sum of the absolute values of the differences. So we have to find positive numbers R_i such that the following three conditions are fulfilled:

i) $$\sum_{i=1}^{n} |R_i - R_i'| \text{ is minimal,}$$

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*) The reader should realize that the meaning of the symbol R_i' differs from that in paragraphs 2 and 3.

ii) $\sum_{i=1}^{n} R_i = 150$ and

iii) the R_i are integers.

It is easy to see that the solution (when no ties are present) is given by the following algorithm:

Algorithm ABSDIF

step 1. Allocate entier(R_i') seats to party i ($i = 1,2,\ldots,n$).
(Then still 150 - $\sum$ entier(R_i') seats are to be allocated.)

step 2. Arrange the parties in order of the values of the fractions R_i' - entier(R_i'), from the highest to the lowest. Allocate the first remaining seat to the party showing the greatest fraction, the second seat to the party which follows, etc., until all remaining seats are allocated.

Principally, this system corresponds to the system introduced in The Netherlands by ROGET [3].

The algorithm sometimes produces the same distribution of seats as R.E. (see Table 5, case 1), sometimes it yields different, very strange results (see Table 5, case 2, party D).

Table 5

Case 1

Party	R'	1st step	2nd step R	System R.E. R
A	10.75	10	(0.75) 11	11
B	5.75	5	(0.75) 6	6
C	2.2	2	2	2
D	1.3	1	1	1
Total	20	18	20	20

Table 5

Case 2

Party	R'	1st step	2nd step R	System R.E. R
A	10.25	10	10	11
B	5.25	5	5	5
C	3.2	3	3	3
D	1.3	1	(0.3) 2	1
Total	20	19	20	20

Another disadvantage of minimizing the absolute differences is demonstrated by the so-called "Alabama Paradox" [2]. Given a distribution R_i' and the corresponding allocation of seats R_i. Now it is decided to increase the total number of representatives with 1 (this is relevant, for instance, for the U.S.A., but not for the Netherlands). When again the seats are allocated, it is possible that a party loses one seat! An example is given in Table 6. Party C loses one seat when the total number of representatives is brought from 100 to 101. This is unacceptable. It can be proved that in the system R.E. this phenomenon cannot occur.

Table 6

Party	R'	R	R'	R
A	45.29	45	45.74	46
B	44.20	44	44.64	45
C	10.51	11	10.62	10
Total	100	100	101	101

b. The second inequality coefficient we consider is the sum of the absolute values of the *relative* differences. The problem may now be formulated as follows: to find positive numbers R_i such that the following three conditions are fulfilled:

i) $\sum_{i=1}^{n} |R_i - R_i'|/R_i' = \text{minimal}$,

ii) $\sum_{i=1}^{n} R_i = 150$ and

iii) the R_i are integers.

It can easily be proved that the solution (when no ties are present) is provided by the following algorithm:

Algorithm ABSRELDIF

step 1. Allocate round(R_i') seats to party i ($i = 1,2,\ldots,n$) (by round(x) we mean the nearest to x integer). If $\sigma = \sum \text{round}(R_i') = 150$ we are ready. If $\sigma < 150$ go to step 2, else go to step 3.

step 2. ($\sigma<150$) For each party compute the increase of the sum of the absolute values of the relative differences caused by *adding* one seat to the number of seats. Allocate one seat to the party which causes the *smallest* increase. Repeat step 2 (the party which gained the last seat again competes) until all (150) seats are allocated.

step 3. ($\sigma>150$) For each party compute the increase of the sum of the absolute values of the relative differences caused by *subtracting* one seat from the number of seats. Take away one seat from the party which causes the *smallest* increase. Repeat step 3 (the party which lost the last seat again competes) until no more than 150 seats are allocated.

This minimization procedure is not the same as R.E.. There are cases showing remarkable differences, like that in Table 7. Perhaps one may prefer the distribution according to the system R.E., but it seems a question of arbitrary political preference.

Table 7

Party	R'	Step 1	Step 2		Step 2		System R.E.
			Increase	R	Increase	R	R
A	12.01	12	0.08160	13	0.08326	14	13
B	2.40	2	0.08333	2		2	3
C	1.40	1	0.14286	1		1	1
D	1.40	1	0.14286	1		1	1
E	1.40	1	0.14286	1		1	1
F	1.39	1	0.15827	1		1	1
Total	20	18		19		20	20

As for a possible generalization of the minimization systems a and b we remark that it can easily be proved that the distributions of seats produced by algorithms ABSDIF and ABSRELDIF also minimize the inequality coefficients $\sum|R_i - R_i'|^\alpha$ and $\sum(|R_i - R_i'|/R_i')^\alpha$, respectively, for any given positive real number α.

c. The third minimization system we will discuss is based on the inequality coefficient by THEIL [6], which is developed within the concepts of information theory. For a concise and brief introduction into this approach we quote THEIL [7,p.521]. The reader should notice tnat in Theil's paper p_i and q_i correspond to our R_i' and R_i (apart from a factor 150), and that $\sum p_i = \sum q_i = 1$.

What is needed in the first place is a measure which describes the degree to which the total is subdivided. A natural and well-known measure, based on concepts from information theory, is the entropy. For the fractions $p_1,\ldots,p_n$ *the entropy is defined as*

(2.1) $$H_p = -\sum_{i=1}^{n} p_i \log p_i$$

which is nonnegative and which vanishes if and only if $p_i = 1$ *for some* i *(and hence* $p_j = 0$ *for* $j \neq i$*). The maximum is* log n, *which is attained when the* p_i*'s are all equal to* 1/n. *Thus, the entropy takes the smallest value when the scene is dominated completely by one group (which amounts to a minimum of "dividedness") and the largest value when all groups are of equal size; and this maximum,* log n, *increases in turn when the number of groups* (n) *increases. We shall exclude the possibility that* H_p *is equal to either limit:*

(2.2) $$0 < H_p < \log n.$$

If $H_p = 0$, *then some* p_i *is equal to one and all others are zero. Clearly, there exists no reasonable way of assigning parliament seats to parties which did not receive any vote at all. If* $H_p = \log n$, *all* n *parties received the same number of votes, and there exists no reasonable way of allocating parliament seats to these parties on another than equal basis.*

But if H_p *satisfies the constraint* (2.2), *as is normally the case, one can conceive of a set of parliamentary fractions* $q_1,\ldots,q_n$ *whose entropy*

(2.3) $$H_q = -\sum_{i=1}^{n} q_i \log q_i$$

differs from H_p. *Obviously, if we fix* H_q *at a pre-assigned level, we still have freedom as to the choice of* $q_1,\ldots,q_n$ *when* $n \geq 2$, *and we would like to choose the* q*'s as closely to the corresponding* p*'s as is possible. Given that the informational concepts* H_p *and* H_q *are used to measure "dividedness", a natural way of specifying "as close as possible" is the following. Consider* $p_1,\ldots,p_n$ *as the prior probabilities of* n *events and imagine that a message arrives which indicates that the odds in favor of these events have changed to the extent that the new (posterior) probabilities are* $q_1,\ldots,q_n$. *The expected information of this message is defined in information theory as*

(2.4) $$I(q:p) = \sum_{i=1}^{n} q_i \log \frac{q_i}{p_i},$$

where q *and* p *on the left stand for* $q_1,\dots,q_n$ *and* $p_1,\dots,p_n$, *respectively. The information expectation is never negative. It vanishes if and only if* $p_i = q_i$ *for each* i, *and it takes larger and larger values when the* p's *and* q's *are pairwise more different.*

This becomes particularly clear when $I(q : p)$ *is expanded according to powers of* $(q_i - p_i)/p_i$. *If we use natural logarithms (which is done throughout this paper) and disregard third and higher powers, the result is*

$$I(q : p) \approx \frac{1}{2} \sum_{i=1}^{n} \frac{(q_i-p_i)^2}{p_i} ,$$

provided that the p's *and* q's *are pairwise sufficiently close to each other (so that the expansions converge). The right-hand side of this approximation is proportional to a chi-square with the* p's *as theoretical probabilities and the* q's *as observed frequencies.*

The information expectation (2.4) *is a measure for the degree to which we are surprised when we are informed that the prior probabilities* $p_1,\dots,p_n$ *are replaced by the posterior probabilities* $q_1,\dots,q_n$. *Interpret now the* p_i's *again as the proportions of the votes cast on the various parties and the* q_i's *as the corresponding proportions of parliamentary seats. Obviously, the electorate ought to be surprised as little as possible.*

Actually, it turns out (see below) that the system R.E. does minimize the sum

$$(*) \qquad \sum_{i=1}^{n} (R_i - R_i')^2/R_i'$$

over all integers $R_1,R_2,\dots,R_n$ which sum up to 150. Now this sum is the same (apart from a factor 1/300) as the above given approximation of $I(q : p)$ by Theil. So we can conclude that the system R.E. in first approximation minimizes the degree to which one is surprised when one is informed that the exact distribution given by $R_1',R_2',\dots,R_n'$ is replaced by $R_1,R_2,\dots,R_n$.

We will show now for a simple case that indeed the system R.E. minimizes the sum (*), from which a general proof is easily constructed. The first step in R.E. is to take $R_i = \mathrm{round}(R_i')$, where $R_i' = (V_i \times 150)/V$ and $V = \sum_{i=1}^n V_i$. Then clearly the sum (*) is minimal, but sometimes the R_i do not sum up to 150. We suppose $\sum R_i = 149$. Then in R.E. the denominator V in R_i' is replaced by a new denominator $V^* < V$ such that the new R_i' after rounding sum up to 150. But replacing the denominator V by V^* is the same as multiplying with the constant factor

$$c = \frac{V}{V^*}\ .$$

Now we observe, firstly, that R_i is increased by one if and only if c satisfies

$$\frac{R_i+0.5}{R_i'} \le c < \frac{R_i+1.5}{R_i'}\ .$$

Secondly, we observe that increasing R_i by one causes an increase of the sum (*) which is given by

$$\frac{(R_i+1-R_i')^2}{R_i'} - \frac{(R_i-R_i')^2}{R_i'} = -2 + 2\,\frac{R_i+0.5}{R_i'}\ .$$

Let $\min\limits_{1\le i\le n} \left(\frac{R_i+0.5}{R_i'}\right)$ be assumed by party i_0 (and by no other party).
It follows that if we choose c such that

$$\frac{R_{i_0}+0.5}{R_{i_0}'} \le c < \min\left\{\frac{R_{i_0}+1.5}{R_{i_0}'}\ ,\ \min_{\substack{1\le i\le n\\ i\ne i_0}} \frac{R_i+0.5}{R_i'}\right\},$$

then multiplying all R_i' by c and rounding has the effect that R_{i_0} will increase by one (so that $\sum R_i = 150$), causing a minimal increase of the sum (*).

Q.E.D.

With respect to the present system it is interesting to know that it minimizes the sum

$$\sum_{i=1}^{n} (R_i - R_i' + 0.5)^2/R_i'.$$

This can easily be proved following the lines of the minimality proof for the system R.E. It seems to be difficult to think of reasons on the ground of which one should prefer this sum to (*).

It should be kept in mind that in the last resort the (very difficult) decision which system is to be preferred, has to be left to the politicians or policy makers. It is the responsibility of the mathematicians to design systems with desired properties (as far as possible), and to analyze the mathematical properties of these systems.

We close this paragraph by a remark about disproportional allocation. Some ten years ago GROSFELD [1] made the suggestion to take squares of the numbers of votes before starting the allocation procedure. The result is a diminishing number of seats for the smaller parties, which was generally felt desirable at that time. The disproportionality works in favour of the greater parties, which may be clear by comparing e.g. 1, 2, 3, 15, 20 with 1, 4, 9, 225, 400.

Later this suggestion has been put into a broad and sophisticated mathematical framework by THEIL [7], who developed a general design of the above disproportionality by

$$q_i = \frac{p_i^a}{\sum_{i=1}^{n} p_i^a},$$

where p_i corresponds to our V_i, q_i is a new starting number R_i' (apart from a factor 150) and a is some positive real number.

The suggestion made by Grosfeld was a = 2. However, there are other possibilities, which may be chosen in all kinds of election and allocation problems. It may be of interest to see how this is done in the IFORS voting system. We therefore quote THEIL again [7, p.519]. By "The square system (1.1)" Theil means Grosfeld's case a = 2.

It is interesting in this connection that the International Federation of Operational Research Societies (IFORS) faces a similar problem but in precisely the opposite direction. The problem is that the national member societies are of very unequal size, so that IFORS could be dominated more or less permanently by the societies of one or two countries if proportional representation based on national membership were applied. To prevent this, a rule has been adopted which gives each member society a number of votes proportional to the square root of the number of members:

$$(1.3) \qquad q_i = p_i^{1/2} / \sum_{j=1}^{n} p_j^{1/2} \qquad i = 1,\ldots,n$$

where n *is the number of member societies,* p_i *the* i^{th} *society's share of total membership, and* q_i *its share in the IFORS voting procedure. Thus, if the* i^{th} *society has four times as many members as the* j^{th}, *the number of its votes is only twice as large.*

The square (1.1) *and the square-root system* (1.3) *are both special cases of*

$$(1.4) \qquad q_i = p_i^{\alpha} / \sum_{j=1}^{n} p_j^{\alpha} \qquad i = 1,\ldots,n.$$

A specification $\alpha > 1$ *serves to raise the size of the larger groups and to reduce that of the smaller groups, whereas* $\alpha < 1$ *has the opposite effect. The latter variant is sometimes used to protect geographical minorities. The IFORS case is one example; the U.S. Senate is another, since it gives equal representation to all* 50 *states independent of their population. Thus, if* p_i *stands for the number of voters of the* i^{th} *state measured as a fraction of the national total and* q_i *for the proportion of Senators allocated to this state* (1/50), (1.4) *applies to this case when we specify* $\alpha = 0$.

There are thus several examples of the representation system (1.4): $\alpha = 0$ *(U.S. Senate),* $\alpha = \frac{1}{2}$ *(IFORS),* $\alpha = 1$ *(proportional representation), and* $\alpha = 2$ *(Mr. Grosfeld's proposal), plus the case* $\alpha = \infty$ *(i.e., allocating all seats to the ruling party), a procedure used in some of the democratically less advanced countries.*

5. THE WEIGHTED VOTE

In the light of the first four lines of HUNTINGTON's paper [2]:

In the absence of any provision for fractional representation in Congress, the constitutional requirement that the number of representatives of each state shall be proportional to the population of that state cannot be carried out exactly;,

we think it to be relevant to cite here Theorem VIII of the first author's doctor's thesis:

Consider a house chosen by democratic elections such that on the basis of proportional representation N *seats* ($N \geq 1$) *are allocated to* n *parties "in a way as fair as possible", in the proportion* $R_1 : R_2 : \ldots : R_n$ ($R_i \in \mathbb{N}$, $\sum R_i = N$). *Suppose that according to the results of the elections the seats should have been divided in the proportion* $R_1' : R_2' : \ldots : R_n'$ ($R_i' \in \mathbb{Q}$, $\sum R_i' = N$). *Then in case of votings in the house it should be strongly recommended to give the voting weight* R_i'/R_i *to the votes of any representative of party* i*, instead of the usual weight* 1.

The great advantage of the weighted vote will be clear: the representation of the votes of the electorate in the house is perfect, as it should be, and independent of the system of allocation of the seats.

In systems with a voting-threshold there are two possibilities to define the voting weight. It can either be based only on the votes of the parties admitted to Parliament, or it can be based on *all* the votes recorded by the electorate. We think that both cases can be defended.

In Table 8 we give the voting weights for both cases in the results of the elections of 1977 for the Second Chamber of Dutch Parliament. We derive from it two examples of the importance of this correction.

a. Assume that for approval of a certain proposal a majority of 2/3 is required. Now let all representatives vote for the proposal, except those of parties 2 and 8. The result is that the proposal is accepted with a majority of 100 votes for and 50 against. If we use the weighting votes in the first case, the number of votes for approval amounts to 99.7433, so that the proposal is to be rejected.

b. In the second case we have the curious phenomenon that parties 2 and 3 together have gained 4148080/8320725 × 100% = 49.852% of the votes of the electorate (which counts for 74.78 seats), whereas they have 77 of the 150 seats in Parliament.

The weighted votes would correct these unfairnesses of the system perfectly.

Table 8

Party	Number of votes V	Number of representatives R_i	First case		Second case	
			R'_i	Voting weight R'_i/R_i	R'_i	Voting weight R'_i/R_i
1	2 813 793	53	51.7084	0.9756	50.7250	0.9571
2	2 655 391	49	48.7974	0.9959	47.8695	0.9769
3	1 492 689	28	27.4308	0.9797	26.9091	0.9610
4	452 423	8	8.3141	1.0393	8.1560	1.0195
5	177 010	3	3.2529	1.0843	3.1910	1.0637
6	143 481	2	2.6367	1.3184	2.5866	1.2933
7	140 910	3	2.5895	0.8632	2.5402	0.8467
8	79 421	1	1.4595	1.4595	1.4317	1.4317
9	77 972	1	1.4329	1.4329	1.4056	1.4056
10	69 914	1	1.2848	1.2848	1.2604	1.2604
11	59 487	1	1.0932	1.0932	1.0724	1.0724
	8 162 491	150	150.0002		147.1475	
others	158 234					
Total	8 320 725					

The system of the weighted vote should be (and can easily be) realized in practice as an automatic system: Each representative will have two buttons at his disposal, one for voting for and one for voting against. The votes recorded by the representatives are summed automatically, each with its proper weight. The result of the voting is displayed at the Chairman's desk almost immediately after the voting. Once in every four years only the weights will have to be changed in this automatic system.

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